THE TWENTY-SIXTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth annual meeting of the Society was held at Columbia University on Tuesday and Wednesday, December 30–31, extending through two sessions on each day. The attendance included the following ninety-six members:

Dr. J. W. Alexander, Mr. P. L. Alger, Professor R. C. Archibald, Professor C. S. Atchison, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor Daniel Buchanan, Professor R. W. Burgess, Professor W. D. Cairns, Miss M. F. Chadbourne, Dr. Teresa Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Professor J. V. DePorte, Professor C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Professor C. H. Forsyth, Mr. T. C. Fry, Professor A. S. Gale, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor H. E. Hawkes, Professor Olive C. Hazlett, Professor L. A. Howland, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. E. A. T. Kircher, Professor P. A. Lambert, Dr. K. W. Lamson, Professor D. D. Leib, Professor Florence P. Lewis, Professor P. H. Linehan, Professor Joseph Lipka, Professor C. R. MacInnes, Professor H. H. Mitchell, Professor C. N. Moore, Professor Frank Morley, Dr. H. C. M. Morse, Professor G. W. Mullins, Mr. F. H. Murray, Professor F. W. Owens, Dr. Helen B. Owens, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Arthur Ranum, Professor H. W. Reddick, Dr. C. N. Reynolds, Jr., Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. W. G. Simon, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Mr. J. J. Tanzola, Dr. J. S. Taylor, Mr. H. M. Terrill, Professor H. D. Thompson, Mr.

H. S. Vandiver, Professor Oswald Veblen, Mr. J. L. Walsh, Mr. H. E. Webb, Dr. Eula A. Weeks, Professor Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Dr. Norbert Wiener, Professor F. B. Wiley, Professor Ruth G. Wood, Professor J. W. Young.

President Frank Morley occupied the chair, being relieved at the last session by Professor J. L. Coolidge. The Council announced the election of the following persons to membership in the Society: Dr. H. E. Bray, Rice Institute; Professor I. L. Miller, Carthage College; Dr. Helen B. Owens, Cornell University; Professor E. W. Pehrson, University of Utah. Ten applications for membership in the Society were received.

The following resolutions, introduced by Professor R. C. Archibald as chairman of the committee on bibliography, were adopted by the Council:

1. The Council regards the preparation and publication, in America, of a dictionary of mathematical terms as not only most desirable but also entirely feasible, provided that financial aid for the preparation of the manuscript can be secured.

2. Impressed with possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: 1. The preparation and publication by societies or individuals of surveys, introductory monographs, translations, memoirs, and treatises, in important fields, including the history of mathematics. 2. The organization of research fellowships. 3. The preparation and publication of an encyclopaedia of mathematics in English. 4. The preparation and publication of an annual critical survey, in English, of the mathematical literature of the world. 5. The preparation and publication of a biographical and bibliographical dictionary of mathematicians.

It was decided to proceed with the incorporation of the Society under the general law of the State of New York. A committee was appointed to consider plans for the organization and administration of the Society after the retirement of the present Secretary and Librarian from their offices at the close of the present year. A committee was also appointed to

consider the formation of an international union of mathematicians. The committee on mathematical requirements presented a report, which was laid over for consideration at the February meeting.

The total membership of the Society is now 733, including 80 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 393; the number of papers read was 187. The number of members attending at least one meeting during the year was 252. At the annual election 156 votes were cast. The Treasurer's report shows a balance of $10,692.23, including the life-membership fund of $7,168.87. Sales of the Society's publications during the year amounted to $1,811.52. The Library now contains 5,690 volumes, excluding some 500 unbound dissertations.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professor C. N. HASKINS,
Professor R. G. D. RICHARDSON.

Secretary, Professor F. N. COLE.

Treasurer, Professor J. H. TANNER.

Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1922,

Professor T. H. HILDEBRANDT, Professor W. A. MANNING,
Professor EDWARD KASNER, Professor H. H. MITCHELL.

The meeting of the Society immediately preceded that of the Mathematical Association of America on January 1–2. A very pleasant occasion was the joint dinner of the two organizations on New Year's eve with an attendance of 114 members and friends.

The following papers were read at the annual meeting:

(1) Dr. H. F. MACNEISH: "The sum of the face angles of a polyhedron in space of n dimensions."

(2) Dr. G. A. PFEIFFER: "A connected set of points which contains no continuous arc."

(3) Professor C. J. KEYSER: "Fundamental types of groups of relations of an infinite field."

(4) Professor JOSEPH LIPKA: "The theorem of Thomson and Tait and its converse in space of n dimensions."

(5) Professor A. A. BENNETT: "Poncelet polygons in higher space."

(6) Professor A. A. BENNETT: "Continuous matrices, algebraic correspondences, and closure."

(7) Professor L. A. HOWLAND: "Concerning points of inflection on a rational plane quartic."

(8) Dr. H. C. M. MORSE: "Geodesics motion on a surface of negative curvature."

(9) Professor J. L. COOLIDGE: "The geometry of hermitian forms."

(10) Professors H. B. PHILLIPS and C. L. E. MOORE: "Rotations in space of even dimensions."

(11) Professors C. L. E. MOORE and H. B. PHILLIPS: "Note on geometric products."

(12) Professor O. E. GLENN: "A memoir upon formal invariancy with regard to binary modular transformations."

(13) Professor O. E. GLENN: "The invariant problem of the relativity transformations of Lorentz appertaining to the mutual attraction of two material points" (preliminary report).

(14) Dr. NORBERT WIENER: "The mean of a functional of arbitrary elements."

(15) Dr. NORBERT WIENER: "Bilinear operations generating all operations rational in a domain Ω."

(16) Dr. NORBERT WIENER: "Fréchet's calcul fonctionnel and analysis situs."

(17) Dr. NORBERT WIENER: "A set of postulates for fields."

(18) Mr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(19) Mr. J. L. WALSH: "On the proof of Cauchy's integral ormula by means of Green's formula."

(20) Professor DUNHAM JACKSON: "On the order of magnitude of the coefficients in trigonometric interpolation."

(21) Mr. P. L. ALGER: "A problem of electrical engineering."

(22) Professor W. B. FITE: "Properties of the solutions of certain functional differential equations."

(23) Professor W. C. GRAUSTEIN: "Determination of the pairs of ordered real points representing a complex point."

(24) Dr. J. S. TAYLOR: "Sheffer's set of five postulates for Boolean algebras in terms of the operation 'rejection' made completely independent."

(25) Professor C. H. FORSYTH: "$_nE_x$, the magic wand of actuarial theory."

(26) Professor C. H. FORSYTH: "A formula for determining the mode of a frequency distribution."

(27) Professor DANIEL BUCHANAN: "Asymptotic orbits near the equilateral triangle equilibrium points in the problem of three finite bodies."

(28) Professor C. L. BOUTON: "The definition of birational transformations by means of differential equations."

(29) Professor W. C. GRAUSTEIN: "Area-preserving, parallel maps in relation to translation surfaces."

(30) Dr. C. N. REYNOLDS, Jr.: "Note on linear differential equations of the fourth order whose solutions satisfy a homogeneous quadratic identity."

(31) Professor J. E. ROWE: "A practical problem of aerodynamics and thermodynamics."

(32) Professor W. B. CARVER and Mrs. E. F. KING: "A property of permutation groups analogous to multiple transitivity."

(33) Professor OLIVE C. HAZLETT: "Some pseudo-finiteness theorems in the general theory of modular covariants."

(34) Dr. MARY F. CURTIS: "Note on the rectifiability of a twisted cubic."

(35) Dr. TERESA COHEN: "The representation of fractions of periods on algebraic curves by means of virtual point sets."

(36) Professor C. A. FISCHER: "Necessary and sufficient conditions that a linear transformation be completely continuous."

(37) Dr. S. D. ZELDIN: "On the structure of finite continuous groups with a single exceptional infinitesimal transformation."

(38) Mr. J. L. WALSH: "On the location of the roots of the derivative of a polynomial."

Dr. Zeldin was introduced by Professor Lipka. The paper of Dr. MacNeish, the first paper of Professor Bennett, the

papers of Professors Phillips and Moore, the third and fourth papers of Dr. Wiener, the first paper of Professor Forsyth, and the paper of Professor Carver and Mrs. King were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. MacNeish obtains the following formula:

$$\Sigma = \left[\tfrac{2}{3}n(n-1)a_0 - \frac{8}{n-2}a_3 \right]$$

right angles for the sum of the face angles of the plane faces of a simple polyhedron in n-space, where a_r ($r = 0, 1, \ldots, n-1$) represents the number of r-space faces. A simple polyhedron in n-space is defined as a polyhedron in which ${}_{n-i}C_{r-i}$ of its r-space elements meet at each i-space ($r > i$; $i = 0, 1, \ldots, n-2$; $r = 1, 2, \ldots, n-1$). If more than ${}_{n-i}C_{r-i}$ of the r-space elements meet at any i-space, the polyhedron is called a multiple polyhedron. Formulas are developed for the sum of the face angles of certain types of multiple polyhedrons.

2. In this paper Dr. Pfeiffer presents an example of a connected set of points which contains no continuous arc. This example is the set of points (x, y) such that

$$y = \sum_1^\infty c_n \phi(x - \omega_n) \qquad (-\tfrac{1}{2} < x < \tfrac{1}{2}),$$

where

$$\phi(x - \omega_n) = \sin \frac{1}{x - \omega_n},$$

$x \neq \omega_n$ and $-1 - \omega_n < x < 1 - \omega_n$, $\phi(0) = 0$; ω_n is the abscissa of the nth rational point of the interval $-\frac{1}{2} < x < \frac{1}{2}$ with respect to a definite enumeration of the rational points of that interval and $c_1, c_2, c_3, \ldots$ is a denumerable set of positive numbers such that $\sum_1^\infty c_n$ is convergent.

3. In Professor Keyser's paper the relations of an infinite field (coincident with the domain and the codomain) are

viewed as falling under nine types: $1-1$, $1-s$, $s-1$, $s-s$, $1-m$, $m-1$, $s-m$, $m-s$, $m-m$, where m means many (i. e., two or more) and s means some in the sense that a relation whose type symbol involves s contains at least one couple in which s is represented by only one term and at least one couple in which s is represented by more than one term. A class of relations containing one or more relations of each of k of the nine types, as $t_1, t_2, \cdots, t_k$, is said to be of the type $t_1 \smile t_2 \smile \cdots \smile t_k$. There thus arise 511 class types, called fundamental types. Among the infinitude of classes of a given type there is one class which includes every subclass of the same type. This distinguished class is said to be a fundamental class. Accordingly there are 511 fundamental classes, one for each fundamental type. Given a rule by which any two relations of the field can be combined so as to yield a relation of the field, there arise two problems: (1) To determine which of the fundamental classes are groups under the rule; (2) To determine which of the fundamental classes contain group subclasses of the same type as that of the containing class. The solution of (1), the rule being that of relative multiplication, was given by the author in a paper presented at the last April meeting of the Society. The present paper gives the solution of (2) for the same rule, and the solutions of both problems where the rule of combination is that of logical addition of relations.

4. The theorem of Thomson and Tait for space of n dimensions may be stated as follows: if from all points of an arbitrary hypersurface (space of $n-1$ dimensions) particles not mutually influencing one another be projected normally in a conservative field of force, points which they reach with equal actions lie on a hypersurface cutting the paths at right angles. The paths form a system of $\infty^{2(n-1)}$ trajectories for each value of the constant of energy. The converse theorem proved by Professor Lipka states: if a system of $\infty^{2(n-1)}$ trajectories in space of n dimensions is such that any ∞^{n-1} trajectories of the system which meet an arbitrary hypersurface (space of $n-1$ dimensions) orthogonally admit of ∞^1 orthogonal hypersurfaces, then the system may be considered as the trajectories in a conservative field of force. The purely geometric part of the theorem is true for all *natural* families of

curves. The n-dimensional spaces considered have constant curvature.*

5. Professor Bennett's first paper appears in full in the present number of the BULLETIN.

6. This paper by Professor Bennett shows how an algebraic correspondence may be viewed as a matrix with a continuous array of elements. The result of two successive correspondences is represented by the matrix obtained with the usual rules for a product. The sum of two matrices has an analogous interpretation in correspondences. The matrix representation emphasizes a weighting of correspondences which is suggested, but less readily, by geometric considerations. The meaning of the inverse of reducible correspondences becomes precise only when weights are assigned in the manner required by the matrix representations. The inverse of a weighted algebraic correspondence may fail to exist altogether or may not be algebraic when these terms are carefully defined. The question of the algebraic character of an inverse that exists is entirely equivalent to the closure of the original correspondence, and to the existence of variable closed algebraic configurations on the fundamental manifold.

7. It is well known that the six inflexions of a rational plane quartic lie on a conic. To the best knowledge of the author, however, there is no proof of this theorem which is at once complete and also based on elementary considerations. Professor Howland's paper attempts such a proof, and follows that by special consideration of the cases of a quartic with a triple point.

8. The study of dynamical systems leads at once to the study of geodesics as a type of motion. A primary purpose of Dr. Morse's study of geodesics was to establish the existence of a class of recurrent motions called discontinuous recurrent motions by Professor Birkhoff, and to examine a case where, if the recurrent motions, not simply periodic, existed at all, they would furnish a very general example of such motions. The surfaces of negative curvature considered are surfaces

* For a discussion of these theorems for space of three dimensions see Edward Kasner, *Trans. Amer. Math. Society*, vol. 11 (1910), pp. 121–140.

which may be cut up into n simply connected portions (n arbitrary, and exceeding one) none of which lie wholly in any finite portion of space, together with one portion S which lies wholly in a finite portion of space, and which may be of arbitrary connectivity exceeding two. Geodesics are considered which if continued indefinitely in either sense lie wholly on a finite portion of the surface. A fundamental result of this paper is a one-to-one representation of these geodesics in terms of the $2p$ fundamental contours of S, together with the n closed curves bounding S and separating S from the remaining portions of the original surface. A second paper will apply the results of this paper to the theory of dynamical systems.

9. In Professor Coolidge's paper the problem of reducing a hermitian form of non-vanishing discriminant is discussed. Special attention is paid to those collineations of n-space which leave invariant the form $\Sigma x_i \bar{x}_i$ ($i = 0, 1, 2, \cdots, n$).

10. Professors Phillips and Moore treat rotations in space of $2n$ dimensions by means of the Gibbs dyadic, deriving in particular a form

$$\phi = e^{I \cdot M} = e^{I \cdot \Sigma q_i M_i}$$

for such a rotation. Here M is a complex two-vector, M_i a set of completely perpendicular plane vectors, and q_i the angles of rotation in those planes, I being the identical dyadic and $I \cdot M$ the dot product of Lewis. A sufficient condition that two rotations ϕ and ϕ' be commutative is that the vector of the dyadic $(I \cdot M)$ $(I \cdot M')$ be zero. This is also necessary unless two or more of the q's equal $\pm \pi$.

In four dimensions any rotation can be expressed as the product of two equiangular rotations

$$e^{qI \cdot (M_1 + M_2)} \quad \text{and} \quad e^{q'I \cdot (M_1 - M_2)}.$$

These equiangular rotations have the property of rotating all vectors through the same angle and are analogous to the rotations resulting from the multiplication of quaternions referred in the first case to right-hand axes, in the second to left-hand ones.

11. Products which in terms of units have identities of the form $k_{12}k_{34} = k_{1234}$ (outer product of Grassmann) and $k_1 k_{12} = k_2$

(inner product) are well known. Professors Moore and Phillips consider such products as $k_{12}k_{13} = k_{23}$, this particular one being an extension of Gibbs' cross product. These occur in the analytic expression of complete perpendicularity and in the study of commutative rotations.

12. The memoir of Professor Glenn comprises a general exposition relating to the following subdivisions: Invariants belonging to prescribed domains of transformations of finite order on n variables. A complete system under the above theory, in the $G\,F(p^2)$, of a binary quantic of order m. Systems of universal covariants of the groups G_x: $x_1 \equiv x_1' + x_2'$, $x_2 \equiv x_2' \pmod{p}$, H_x: $x_1 \equiv x_1' + x_2' + x_3'$, $x_3 \equiv x_3' \pmod{p}$, and of the simultaneous groups G_x, G_y. A theorem upon modular systems according to which the formal concomitants modulo p of a binary m-ic are constructed as simultaneous systems of sets of covariants of lower order. Development of a fundamental system of covariants for a linear form and a quadratic modulo 2, of a single binary quartic modulo 2, and of a complete system of semivariants of the quartic modulo 3. Somewhat extensive data concerning a complete system of covariants of the quartic modulo 3 are also included. The last section of the paper gives a theory of complete systems under the transformations on the velocities and accelerations which figure in the theory of relativity of motion.

13. The components F_x, F_y, F_z in the theory of the mutual attractions of two material points are invariantive under the linear transformations S on four variables, called by Lorentz the general transformations of relativity (time being a fourth dimension). A theory of concomitants in the domain of the coefficients of S, based upon the equations of the four planes upon its poles, is treated in Professor Glenn's paper.

14. Dr. Wiener shows how, by applying to E. H. Moore's notion of development the notion of a weighting (i.e., of a measure of volume), a type of mean can be obtained which, under certain very general hypotheses, will be an example of the generalized integration of P. J. Daniell. The means of a function of n variables and of a function of a discrete infinity of variables are shown to be examples of this concept, while it is proved that by a proper "weighting" the mean or

integral of a function of a line can be defined. In this connection, the concept of the "coefficient of irregularity" of a function is introduced, viz.,

$$\operatorname{Max} \frac{\Delta f(x)}{\Delta(x)},$$

and it is proved that a bounded set of functions whose coefficients of irregularity are all less than some finite quantity is extremal in the sense of Fréchet.

15. An operation $f(x_1, x_2, \ldots, x_p)$ is said to generate a class K of operations when

(1) K contains f;

(2) if K contains $g(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k)$, it also contains $g(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k)$ and $g(x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k)$.

(3) if K contains $g(x_1, x_2, \ldots, x_k)$ and $h(x_1', x_2', \ldots, x_l')$, K also contains the operation on $x_1, x_2, \ldots, x_{i-1}, x_{i+1} \ldots, x_k, x_1', x_2', \ldots, x_l'$ formed by substituting h for x_i in g.

(4) K contains no proper subset with the three preceding properties.

Dr. Wiener proves that the operations that are bilinear and generate the set of all rational operations with rational coefficients may be reduced by a rational linear transformation to one of the forms

$$\frac{x - y}{x + Gy} \quad (G \text{ rational and } \neq -1)$$

$$\frac{n(x - y + xy)}{ny + xy} \quad (n \text{ integral and } \neq 0),$$

and that every such operation will generate all rational operations with rational coefficients. The results are extended in part to other domains of rationality.

16. Dr. Wiener defines continuous deformations and their limits on the basis of the limit of a sequence of points. The notion of a one-parameter family of deformations follows readily, and an n-space can be defined by employing this concept together with a process of mathematical induction.

Certain properties of the n-spaces selected from a given space are discussed, and finally a categorical set of postulates is given for the analysis situs of a space of any finite number of dimensions.

17. Dr. Wiener shows that all algebraic operations are obtainable by the iteration of $x|y = 1 - x/y$. A set of postulates is derived for this operation. This set is consistent, independent, and equivalent to the sets given by E. V. Huntington for fields. It is also shown that every operation and concept in complex algebra can be derived from the operation $1 - K(x/y)$, where K stands for "conjugate of."

18. Mr. Walsh's paper is a continuation of a recent paper published in the *Transactions* under the same title. It is proved as a lemma that if three variable points of the complex plane determine a fourth variable point by a real constant cross ratio with those three, and if the three points have as their respective envelopes closed regions bounded by circles, then the envelope of the fourth point is a fourth closed region bounded by a circle. This lemma is applied in proving that if three circular regions contain respectively k roots of a binary form of degree p_1, the remaining $p_1 - k$ roots of this form, and all the roots of a second form of degree p_2, then these three circular regions and the fourth circular region of the lemma corresponding to the cross ratio p_1/k contain all the roots of the jacobian of the two forms. If the four regions have no common point, they contain respectively the following numbers of roots of the jacobian: $k - 1$, $p_1 - k - 1$, $p_2 - 1$, and 1.

19. Mr. Walsh's second paper appeared in full in the January BULLETIN.

20. It is well known that there is a remarkable and far-reaching parallelism between the properties of the trigonometric interpolating formula determined by the values of a given function at a set of equally spaced points, and those of the formula of approximation obtained by taking the first terms of the Fourier's series for the same function. This parallelism is closer than would be indicated at first sight by the correspondence of sums and integrals in the formulas for the coefficient of the two approximating functions. It has

to be regarded as due in part to the special properties of the trigonometric functions, not merely to the definition of an integral as the limit of a sum. Professor Jackson's paper is concerned with the order of magnitude of the coefficients in the interpolating formula, in case the given function is of limited variation, or has a derivative of limited variation. The method of partial summation is used to give results analogous to those which Picard (Traité d'Analyse, volume 1) obtains for the Fourier's series by the second law of the mean and integration by parts. Incidentally, in the case of a derivative of limited variation, a relatively simple proof is obtained for the convergence of the Fourier's series, as well as of the interpolating formula, to the value of the given function.

21. Mr. Alger presents an unsolved problem of electrical engineering which is of considerable practical interest to engineers. The problem is to determine the distribution of magnetic flux and current density across the thickness of an infinite plane lamina of iron, when an harmonically varying magnetizing force is applied. Under the assumption of uniform permeability, the problem has been completely solved, and its solution may be found in any standard advanced text book of electrical engineering. The solution is in this case analogous to that of the flow of heat in a plane, and the flux and current densities at any point are proportional to the hyperbolic functions of a complex quantity. In actual practice, however, the permeability varies over a wide range and is directly dependent on the magnetizing force. The solution of the problem, when the permeability is assumed variable, has never been obtained, and the whole problem under this assumption is so little understood that the simplest qualitative results of a mathematical study of it should prove valuable. When the permeability is variable, the harmonic impressed magnetizing force sets up magnetic fluxes of all odd multiples of the impressed frequency, and the solution of the problem involves the determination of the coefficients in a Fourier's series. The law of variation of permeability is a purely empirical one, so that one difficulty in the problem is the choice of a law of variation that will make a solution possible while it still sufficiently represents the facts.

22. In this paper Professor Fite discusses the oscillatory properties of the solutions of binary functional differential equations of the form $y^{(n)}(x) + r(x)y(k - x) = 0$, where $|r(x)| > b > 0$ for all finite values of x. He shows that when n is odd every solution changes sign an infinite number of times, but that when n is even a solution may change sign only a finite number of times. Whether this finite number is odd or even depends upon the sign of $r(x)$. He also determines certain necessary conditions in order that these equations, for $n \leqq 2$, may have odd or even solutions.

23. A complex point P in a complex space of n dimensions, since it depends on $2n$ real parameters, may be represented by a pair of real points P_1, P_2 in the space. It is reasonable to restrict the choice of P_1, P_2 so that (*a*) they coincide in P, if P is a real point; (*b*) they are the same for the point $\overline{P}$, conjugate to P, as they are for P, and in one order represent P and in the opposite order $\overline{P}$; (*c*) their coordinates, referred to a cartesian system, are analytic functions of the coordinates of P and $\overline{P}$, referred to this system; (*d*) their transforms P_1', P_2' by an arbitrary transformation of a chosen group of real point transformations represent the point P′ into which P is carried by the transformation. Professor Graustein determines, for the group of real collineations and for each of the more important of its subgroups, the affine group, the group of similarity transformations, the group of motions and the group of translations, all the ordered point pairs P_1, P_2, which represent the given point P and have the desired properties. The determination, in the case of each group, depends on the solution of a functional equation.

24. As has been shown by L. L. Dines, Sheffer's five postulates for Boolean algebras in terms of the operation "rejection," while independent in the sense that no one of them is implied by the other four, are not completely independent in the sense defined by Professor E. H. Moore, in as much as the negative of the first postulate implies the third, fourth, and fifth. Dr. Taylor's paper demonstrates the fact that the postulates become completely independent if the first postulate is replaced by one postulating a minimum of four distinct elements instead of two.

25. The present value of a pure endowment, or ${}_nE_x$, seems to be generally regarded as on a par in importance with quite a number of other actuarial symbols of similar nature. Professor Forsyth shows how the symbol may be used to affect ready analyses of numerous situations in actuarial theory which otherwise prove subtly complicated. In particular, he gives concise derivations of formulas used in present day methods of valuation of life insurance policies—such as full preliminary term, modified preliminary term, etc.—wherein the conciseness is due directly to the use of the symbol ${}_nE_x$.

26. The main disadvantage in the use of the mode as an average is the difficulty met in its determination when the statistical data are given in the form of a frequency distribution when the frequencies correspond to class-marks. Professor Forsyth derives a formula for determining the mode in situations like those just mentioned, based upon his formula used for interpolation of ordinates among areas presented in the December (1916) issue of the *Quarterly Publications of the American Statistical Association.*

27. In a previous paper, presented to the Society in December, 1918, Professor Buchanan discussed the orbits which are asymptotic to the straight line equilibrium points in the problem of three finite bodies. The present paper deals with the corresponding orbits for the equilateral triangle equilibrium points.

28. This note shows how to set up certain systems of partial differential equations whose solutions establish all the birational transformations in space of n dimensions. Let $x_1, x_2, \cdots, x_{n+1}$ be homogeneous coordinates. The transformation $\rho x_i' = \varphi_i(x_1, x_2, \cdots, x_{n+1})$ $(i = 1, 2, \cdots, n+1)$ is birational if the functions φ_i are homogeneous polynomials of the same degree, m, and if the inverse transformation has the same form, its polynomials being of degree m'. Professor Bouton begins by reducing the transformation by means of projections to a normal form in which $\varphi_1 = x_1^m + x_1^{m-2}A_{12} + \cdots + A_{1m}$, $\varphi_i = x_i x_1^{m-1} + x_1^{m-2}A_{i2} + \cdots + A_{im}$ $(i = 2, 3, \cdots, n+1)$, where A_{ik} is a homogeneous polynomial in $x_2, \cdots, x_{n+1}$, of degree k, and with the inverse in similar form. The differential equations established are those satisfied by

the polynomials A_{ik}. For example, if $m = m' = 2$, the A_{i2} are solutions of the system of partial differential equations

$$2x_iA_{12} + \sum_{s=2}^{n+1} A_{s2}\frac{\partial A_{i2}}{\partial x_s} = 0 \qquad (i = 2, 3, \cdots, n+1).$$

Conversely, any set of solutions of this system, which are homogeneous polynomials of degree two, will yield a birational transformation, which may, however, be reducible to a projection. There seem to be no references in the literature to this method of setting up all the birational transformations in space of n dimensions.

29. Two real surfaces, in continuous one-to-one point correspondence, with the directed normals in corresponding points parallel, may be said to correspond by a parallel map, or more specifically, by a directly parallel or an inversely parallel map, according as corresponding directions of rotation about corresponding points are the same or opposite. Professor Graustein shows that two real surfaces, which correspond by an inversely parallel, area-preserving map of a particular kind, are translation surfaces with real generators, whereas, if the map is directly parallel and area-preserving of a certain type, the surfaces are translation surfaces with conjugate-imaginary generators. In particular, two real surfaces, applicable by an inversely parallel map, are translation surfaces with real perpendicular generators and hence cylinders; if, on the other hand, the surfaces are applicable by a directly parallel map with the property that the angle between corresponding curves in corresponding points is constant, not 0 or π, they are translation surfaces with conjugate-imaginary minimal curves as generators and are, then, minimal surfaces.

30. Dr. Reynolds proves that if a given linear differential equation of the fourth order has four linearly independent real solutions which satisfy a non-singular quadratic identity, then it may be reduced to a form which may be said to be self-adjoint with respect to the third row of the wronskian of any four linearly independent solutions. The signature of the quadratic identity associated with the equation is then evaluated in terms of the coefficients of the given equation and several theorems concerning the zeros of such an equation are proved.

31. The purpose of Professor Rowe's paper was to make a mathematical statement of the aeronautical radiator problem.

32. Professor Carver and Mrs. King define a property of permutation groups, closely analogous to multiple transitivity, as follows:

A permutation group of degree n is said to be k-ply pseudo-transitive, $1 \leqq k \leqq n-1$, if there is a set of k letters such that there is at least one permutation of the group sending this set into any arbitrarily chosen set of k letters *in some order*. It is evident that the case $k = 1$ is the same as ordinary simple transitivity; that if a group is k-ply transitive it is k-ply pseudo-transitive, but not conversely; and that if a group of degree n is k-ply pseudo-transitive it is also $(n-k)$-ply pseudo-transitive. Several theorems analogous to those for ordinary multiple transitivity are readily obtained.

Among the groups of degree equal to or less than nine, there are just seven cases of groups having multiple pseudo-transitivity of a multiplicity other than is obviously implied by their ordinary multiple transitivity.

The notion has an incidental application to, and was suggested by, a question concerning the condition for apolarity between two binary forms.

33. In any theory of covariants, we find that closely related to the point of view taken when we try to express all covariants as polynomials in a finite number of covariants is the one taken when we endeavor to express all covariants as rational (though not necessarily integral) functions of a finite number of covariants. Professor Hazlett's paper attacks modular covariants of a single binary form from this second point of view by employing a method which has its origin in Hermite's fundamental memoir on associated forms. The main theorem of the present paper proves that, if f is a binary form of order not congruent to zero modulo p, then any modular covariant of f for the $GF[p^n]$ is expressible (aside from a power of f) as a polynomial in Q and L, where the coefficients of the terms in Q are polynomials in the forms associated with f. From this theorem flow several corollaries, of which one gives a neat method of constructing a modular covariant having a given leader. This theorem, together with the corollaries, is verified for the binary quadratic, modulo 3.

34. Dr. Curtis's paper appears in full in the present number of the BULLETIN.

35. Suppose a curve C of order m and genus p with d double points. Miss Cohen considers the linear series of curves of order αn cutting C n times at each of $\alpha m - p$ fixed points, among which are the d double points, the value of α for m odd being $\frac{1}{2}(m-1)$, and for m even, $\frac{1}{2}(m-2)$. There are exactly n^{2p} such curves cutting C n times at p further points. Suppose such a set of p points to be $A_1, A_2, \cdots, A_p$, the fixed base points to be $F_1, F_2, \cdots, F_{\alpha m-p}$, and any line section represented by L. Then $A_1 + \cdots + A_p + F_1 + \cdots + F_{\alpha m-p} - \alpha L$ is a virtual set that has the properties of an nth of a period for a given set of $\alpha m - p$ base points and the totality of such sets can be shown to represent the totality of $n$ths of periods. For p sufficiently high it is possible to put a curve of order α on $F_1, \cdots, F_{\alpha m-p}$ with p residual intersections $P_1, P_2, \cdots, P_p$. The nth of a period can then be represented by $A_1 + \cdots + A_p - (P_1 + \cdots + P_p)$.

The simplest case is presented by the elliptic parameters on the general cubic.

36. F. Riesz has derived a large part of the Fredholm theory of integral equations for the equation

$$h(x) = f(x) + \lambda T(f),$$

where $T(f)$ is a completely continuous, linear transformation, and he has also proved that every linear transformation, that is to say every linear functional depending on a parameter, can be expressed in terms of a Stieltjes integral such as

$$T(f) = \int_a^b f(y) d_y K(x, y).$$

In the present paper Professor Fischer has proved that if such a transformation is to be completely continuous, it is necessary and sufficient that the variation of $K(x, y)$ in y shall be a bounded function of x, and that whenever $x_1, x_2, \cdots$ are chosen in such a way that $K(x_n, y)$ approaches a unique function of y when n becomes infinite, the variation of $[K(x_n, y) - \text{limit } K(x_n, y)]$ in y shall approach zero. Another necessary and sufficient condition is also found and proved equivalent to this one.

37. The problem of the structure of finite continuous groups consists in finding the structural constants determined by the alternants of operators, generating a group with r essential parameters, of the form

$$X_i = \sum_{k=1}^{n} \xi_k(x) \frac{\partial}{\partial x_k} \qquad (i = 1, 2, \cdots, r).$$

These alternants have the form

$$(X_i, X_j) = \sum_{s=1}^{r} c_{ijs} X_s \qquad (i, j = 1, 2, \cdots, r).$$

The c's which are the structural constants must satisfy two conditions:

(1) $$c_{ijk} = -c_{jik},$$

(2) $$\sum_{s=1}^{r} (c_{ijk}c_{kls} + c_{jlk}c_{kis} + c_{lik}c_{kjs}) = 0 \quad (k = 1, 2, \cdots, r).$$

Dr. Zeldin discusses the conditions to be imposed on the group with one exceptional infinitesimal transformation X_r, which would make all c_{irk} equal to zero, where $(i, k = 1, 2, \cdots, r)$.

38. Jensen has recently stated the theorem that if $f(z)$ is a real polynomial and if there are described circles having as diameters the segments joining the pairs of conjugate imaginary roots of $f(z)$, then no non-real root of $f'(z)$ lies outside those circles. Mr. Walsh shows that on any segment of the axis of reals exterior to all of those circles and containing no root of $f(z)$ there is at most one root of $f'(z)$. If any of those circles has on or within it k roots of $f(z)$ and is not interior to nor has a point in common with any other of those circles, it has on or within it not more than $k + 1$ nor less than $k - 1$ roots of $f'(z)$. The paper contains also various other theorems connected with Jensen's theorem.

F. N. COLE,
Secretary.

THE ST. LOUIS MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE thirteenth western meeting of the American Mathematical Society, combining the forty-fourth regular meeting of the Chicago Section, and the twelfth regular meeting of the Southwestern Section, was held at the Soldan High School, St. Louis, Missouri, on Tuesday and Wednesday, December 30 and 31. On Tuesday afternoon there was a joint session with Section A of the American Association for the Advancement of Science, and the Missouri Section of the Mathematical Association of America.

The attendance at this meeting included fifty persons, among whom were the following forty-four members of the society:

Professor N. Altshiller-Court, Professor C. H. Ashton, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor H. Blumberg, Professor W. C. Brenke, Dr. C. C. Camp, Professor A. B. Coble, Professor S. C. Davisson, Professor E. L. Dodd, Professor A. Dresden, Professor O. Dunkel, Professor H. J. Ettlinger, Professor Tomlinson Fort, Dr. J. B. Glasgow, Professor H. Halperin, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor G. O. James, Professor H. E. Jordan, Professor O. D. Kellogg, Professor Solomon Lefschetz, Professor A. C. Lunn, Dr. G. F. McEwen, Professor J. V. McKelvey, Professor T. E. Mason, Professor G. A. Miller, Professor B. E. Mitchell, Professor U. G. Mitchell, Professor F. R. Moulton, Dr. J. R. Musselman, Dr. C. A. Nelson, Professor B. L. Newkirk, Professor P. R. Rider, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Mr. J. B. Rosenbach, Dr. G. W. Smith, Professor R. B. Stone, Professor E. B. Stouffer, Dr. E. H. Taylor, Professor W. D. A. Westfall, Professor K. P. Williams.

At the dinner at the American Annex Hotel on Tuesday evening forty-eight persons were present.

At the business meeting of the Chicago Section and the Southwestern Section held at the end of the morning session of Wednesday, the following action was taken:

It was decided to hold the next meeting of the Southwestern Section at the University of Nebraska on November 27. The

following programme committee was elected: W. C. Brenke, chairman, H. L. Rietz, O. D. Kellogg, secretary.

The Chicago Section elected for the ensuing biennium, Professor R. D. Carmichael, chairman; Professor Arnold Dresden, secretary; Professor C. N. Moore, third member of programme committee.

A resolution thanking Professor E. J. Moulton for his services as acting secretary of the Chicago Section during 1918–19 was unanimously passed.

A resolution on the death of Professor Laenas Gifford Weld on November 28, 1919, was adopted by a rising vote.

The Chicago Section and the Southwestern Section passed a resolution expressing interest in an investigation of the possibility of cooperative work in pure mathematics and in the domain between mathematics and the related sciences, as suggested by the National Research Council.

At the joint meeting on Tuesday afternoon, at which Professor O. D. Kellogg, vice-president of Section A, presided, the following papers were read:

I. Professor G. D. BIRKHOFF, retiring vice-president of Section A of the American Association for the Advancement of Science, "Recent progress in dynamics."

II. Professor G. A. BLISS, retiring chairman of the Chicago Section of the American Mathematical Society, "Some recent developments in the calculus of variations."

III. Mr. H. H. PLATT, "A suggestion for the utilization of atmospheric molecular energy."

The sessions of Tuesday forenoon, Wednesday forenoon and afternoon were presided over by Professors Bliss, Birkhoff and Roever respectively. At these sessions the following papers were read:

(1) Professor C. E. LOVE: "Note on a class of singular integral equations."

(2) Dr. R. F. BORDEN: "On the adjoint of a certain mixed equation."

(3) Professor G. O. JAMES: "On a philosophical aspect of the relativity theory of gravitation."

(4) Professor O. SCHMIEDEL: "On the early history of the number concept."

(5) Mr. B. Z. LINFIELD: "On the relation of the roots of a polynomial to the roots of its derivative."

(6) Professor E. L. DODD: "A comparison of the arithmetic mean with other weighted means under certain laws of error."

(7) Professor S. LEFSCHETZ: "Real abelian varieties."

(8) Professor N. ALTSHILLER-COURT: "On a pencil of nodal cubics."

(9) Professor N. ALTSHILLER-COURT: "On an orthocentric quadrilateral."

(10) Professor G. A. MILLER: "Properties of the subgroups of an abelian prime power group which are conjugate under the group of isomorphisms."

(11) Professor G. A. MILLER: "Groups generated by two operators of order three whose product is of order four."

(12) Professor H. L. RIETZ: "Urn schemata as a basis for the development of correlation theory."

(13) Professor K. P. WILLIAMS: "Concerning a problem in the flow of heat."

(14) Professor TOMLINSON FORT: "Note on Dirichlet and factorial series."

(15) Professor H. J. ETTLINGER: "Boundary value problems with regular singular points."

(16) Professor H. J. ETTLINGER: "Extension of an existence theorem for a non-self-adjoint boundary value problem."

(17) Professor G. C. EVANS: "Divergent Fourier series and integral equations."

(18) Mr. L. L. STEIMLEY: "On the general class of series of the form $Y(x) = c_0 + \Sigma c_n g(nx)$."

(19) Professor A. C. LUNN: "A principle of duality in thermodynamics."

(20) Professor R. L. BORGER: "On the Cauchy-Goursat theorem."

(21) Professor A. B. COBLE: "Some irrational and transcendental aspects of the invariants of a ternary quartic."

(22) Professor A. B. COBLE: "The Cayley symmetroid and the abelian modular functions of genus four."

(23) Professor E. J. WILCZYNSKI: "One-parameter families and nets of ruled surfaces and a new theory of congruences."

(24) Professor A. EMCH: "Intersections of straight lines and systems of plane algebraic curves and the surfaces generated by them."

(25) Professor PIERRE BOUTROUX: "A family of multiform functions."

(26) Dr. G. F. MCEWEN: "Fluid temperature distributions

deduced from sources, boundary conditions, fluid velocity and convection."

(27) Dr. G. F. McEwen: "Direct application of the idea of "random sampling" in estimating the precision of averages, and the significance of differences."

Dr. Borden was introduced by Professor Carmichael. The papers of Professor Love, Dr. Borden, Professor Schmiedel, Mr. Linfield, Professor Evans, Mr. Steimley, Professor Borger, Professor Wilczynski, Professor Emch, and Professor Boutroux were read by title. Abstracts of the papers follow below.

1. Professor Love's paper develops the elementary theory of the equation

$$\varphi(x) = \frac{f(x)}{x^\alpha} + \lambda \int_a^\infty \frac{K(x,t)\varphi(t)}{x^\alpha t^\beta}\, dt,$$

where $\varphi(x)$ is the unknown function, under the assumption that the functions $f(x)$ and $K(x, t)$ are bounded for $x \geqq a$, $t \geqq a$. This case possesses special interest on account of the fact that it can be solved by a direct extension of the method of Fredholm.

2. Dr. Borden considers with the equation

$$\Delta f'(x) + a(x)f'(x) + c(x)f(x) = 0$$

an adjoint equation. The invariants of the two equations under the group $f(x) = v(x)g(x)$ are compared, and also the invariants of the successive transforms of the two equations by the Laplace-Poisson transformations. (Poisson, *Journal de l'Ecole Polytechnique*, volume 6 (1806), pages 127–141.) Conditions are obtained for solutions of both equations in finite form and some remarks are made concerning the self-adjoint case.

3. The success of Einstein's theory of gravitation in removing the outstanding difference between theory and observation in the secular motion of the perihelion of Mercury, and its applicability to other problems of dynamical astronomy are used by Professor James as an argument in support of Bergson's thesis that there is no intrinsic order in nature and that the apparent mathematical order of the external universe has been imposed thereon by the intellect.

4. In this paper Professor Schmiedel contends that the first appearance of the idea of extension of the number concept is found in the Ahmes papyrus.

5. Maxime Bôcher (*Annals of Mathematics*, 1892) gave a geometrical proof of the theorem of Lucas (*Journal de l'Ecole Polytechnique*, 1879), that the roots of the derivative of a cubic are the foci of the maximum ellipse inscribed in the triangle, whose vertices are the roots of the cubic, and asked the question "Could not the roots of the derivative of the general polynomial be brought into connection with the foci of the higher plane curves?" Mr. Linfield answers the question by proving that the roots of the derivative of a polynomial of degree n are the foci of a curve of class $n-1$ which touches the segments joining the roots of the polynomial, two and two, at their midpoints. He generalizes and proves that the roots of the derivative $f'(z)$ of the function $f(z) = \Pi(z-z_r)^{\mu_r}$ are the foci of a curve of class $n-1$ which touches the segment z_iz_j at a point dividing it in the ratio $\mu_i : \mu_j$.

6. Professor Dodd finds that if the probability that the error of a measurement will lie between x and $x+dx$ is $c(1+|hx|^{3/2})^{-1}dx$, with c and h constant, then the arithmetic mean of two measurements is the least probable value among all possible weighted means. Under the law $c_1(1+h^2x^2)^{-1}$, however, the probability that any weighted mean, including the arithmetic mean, will differ from the true value by less than ϵ, is a constant, alike independent of the system of weights and the number of measurements. But under the laws $c_2[(1+h^2x^2)(1+k^2x^2)]^{-1}$, h and k equal or unequal, and $c_3(e^{hx}+e^{-hx})^{-1}$, the arithmetic mean is more reliable than any other weighted mean. The curves representing the foregoing laws of error closely resemble curves of the family $y = c_4e^{-h^2x^2}$, the probability function. The possibility of the arithmetic mean being a least probable value is associated with the fact that under certain laws of error the arithmetic mean of measurements is not as reliable as a single measurement. With an increase of the number of measurements, there is an increase in the probability that the larger errors will appear; and this may more than offset the neutralizing effect of errors of opposite sign.

7. In this note, Professor Lefschetz shows how the number of algebraically distinct real hypersurfaces belonging to real abelian varieties may be obtained from a certain period matrix which characterizes them. This is followed by the actual determination of the number in question for a large class of varieties defined by the possession of certain complex multiplications.

8. Professor Altshiller-Court's first paper appeared in full in the February BULLETIN.

9. The three vertices of a triangle and its orthocenter are the four vertices of an orthocentric quadrilateral having the property, pointed out by Carnot, that each vertex is the orthocenter of the triangle determined by the other three vertices. The four triangles thus formed, often referred to as an orthocentric group of triangles, have the same orthic triangle, and consequently the same nine-point circle, which may therefore be considered as belonging to the quadrilateral itself. Professor Altshiller-Court shows that the four circumcenters of an orthocentric group of triangles form another orthocentric quadrilateral, the two quadrilaterals having the same nine-point circle, the center of which is a center of symmetry of the two orthocentric groups. The four centroids of the given orthocentric group of triangles also form an orthocentric group, the two groups having the same nine-point center, which point is a center of similitude of the two orthocentric groups, the ratio of similitude being -3.

10. It is well known that if an abelian group G has a subgroup of a given type, it has also a quotient group of this type and vice versa. If G is the abelian group of order p^m and of type $(m_1, m_2, \ldots m_\lambda)$ two of its subgroups H_1 and H_2 may be said to be of complementary types if each of them is simply isomorphic with the quotient group of G with respect to the other. A necessary and sufficient condition that two subgroups of types $(m_1', m_2', \ldots, m_\lambda')$ and $(m_1'', m_2'', \ldots, m_\lambda'')$ are complementary is that each of the following equations is satisfied: $m_1' + m_1'' = m_1$, $m_2' + m_2'' = m_2$, $\ldots$, $m_\lambda' + m_\lambda'' = m_\lambda$. The number of the different types of subgroups which are complementary to the subgroups of a given type is equal to the number of the different sets of I-conjugate sub-

groups of G which are simply isomorphic with the subgroups of the given type, I-conjugate subgroups of G being subgroups which are conjugate under its group of isomorphisms. The main object of Professor Miller's paper is to establish some fundamental reciprocal properties of the sets of I-conjugate subgroups contained in certain types of abelian groups of order p^m. Among the theorems established is the following: If all the invariants of an abelian group G of order p^m are equal to each other, then the number of its subgroups of a given type is equal to the number of its subgroups which are simply isomorphic with the quotient group of G with respect to one of the former subgroups. In particular, the number of the distinct subgroups of order p^a contained in G is equal to the number of its distinct subgroups of index p^a.

11. It is known that two operators s_1, s_2 of order 3 whose product is of order 4 generate a group of infinite order unless these operators are otherwise restricted. In the present paper, Professor Miller considers the groups generated by s_1, s_2 when the conditions imposed on these operators are as follows: $s_1^3 = s_2^3 = (s_1 s_2)^4 = (s_1^2 s_2)^k = 1$, $k = 3, 4, 5$. When $k = 3$, these operators generate the group of order 48 obtained by extending the direct product of two cyclic groups of order 4 by means of an operator of order 3 which transforms this direct product into itself but is not commutative with any of its operators besides identity. When $k = 4$, the group generated by these operators is the simple group of order 168 and a new and useful definition of this important group is thus obtained. Finally, when $k = 5$, these operators generate a group of order 1,080 which is isomorphic with the simple group of order 360 with respect to its central. This group of order 1,080 does not contain the simple group of order 360 as a subgroup, but to each operator of the central quotient group there corresponds at least one operator of the group whose order is equal to the order of the corresponding operator in this quotient group.

12. In the present paper, Professor Rietz generalizes certain results of his first paper on this subject. In particular, the urn schema of the first paper that gives t/s for the correlation coefficient, where s is the number of balls in each of a pair of drawings, and t the number of balls taken at random

from the first drawing of a pair to be common to the pair, involves the hypothesis that the urn should be so maintained that the probability of drawing a white ball is 1/2. This restriction is removed in the present paper so that the probability of drawing a white ball may be any constant p. The results are as simple as when $p = 1/2$, and the proofs are but slightly more complicated.

13. A classic problem in the mathematical theory of the flow of heat is to determine the temperature of a semi-infinite solid, initially at the temperature zero, whose plane face is kept at a temperature that varies in a known manner with the time. The solution usually given is that obtained by Duhamel and followed by Riemann. It treats the problem as a limiting case of a simpler one. In the method used by Professor Williams, a definite integral is assumed for a solution. It is suggested by the solution of a related problem on noticing an interchange of boundary conditions. The integrand and limits contain arbitrary elements. These are determined by substituting the assumed solution in the differential equation that it must satisfy.

14. In this note Professor Fort exhibits a series which includes both Dirichlet and factorial series as special cases. He proves a fundamental convergence theorem.

15. The recent study of boundary value problems has been confined to the case where the coefficients of the differential equation are continuous throughout the interval of definition. Except for Porter's and Bôcher's work on the solutions of a differential equation having regular singular points (Bôcher, "Oscillation theorems of Sturm," this BULLETIN, volume 5 (1898)), no attention has been given to the general theory of existence of solutions, oscillation theorems and expansions in normal functions for a system such as occurs in applied problems involving Legendre's, Bessel's or Lamé's differential equations. In this paper, Professor Ettlinger shows that a number of the theorems which hold for the continuous system go over for the system with regular singular points.

16. In a paper to appear in the June number of the *Annals of Mathematics*, Professor Ettlinger establishes the existence of

at least one real characteristic number for the system

$$\frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0,$$

$$A_{i1}u(a) - A_{i2}K(a)u_x(a) - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0,$$

$$(i = 1, 2),$$

where the A_{ij} are constants and certain conditions are satisfied by K and G. This theorem is now extended to a system of the same form where A_{ij} (λ) replaces the constant coefficients of the boundary conditions, subject to the additional restriction that $(A_{1i}A_{2j} - A_{2i}A_{1j})/(A_{1k}A_{2l} - A_{2k}A_{1l})$ decrease (or do not increase) as λ increases, for specified values of i, j, k, l.

17. Professor Evans's paper is an endeavor to connect by practical calculation the concepts of divergent series, integral equation, and Stieltjes integral. The equation treated is equation (1) in Lecture V of the author's Colloquium Lectures, when the kernel is of the form $f(x+s) + g(x-s)$, with f and g periodic with period ab as functions of a single argument. The solution is developed in Fourier series. An "imaginary" resolvent kernel is obtained in the form of a Fourier series, generally divergent and not even summable, which nevertheless yields with proper choice of the known function a summable series as solution of the equation, and thus affords a practical method of handling such equations. The equation treated, which is of the first kind, includes the equation of the second kind as a particular case.

18. In this paper, Mr. Steimley considers the general class of series of the form

$$Y(x) = c_0 + \sum_{n=1}^{\infty} c_n g(nx),$$

where $g(x)$ has the asymptotic character

$$g(x) \sim x^{P(x)}e^{Q(x)}\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

which is valid for x approaching infinity in some sector V bounded by two rays extending from zero to infinity. $P(x)$

and $Q(x)$ are polynomials in x, a_i $(i = 1, 2, \cdots)$ are functions of the angle θ, which is the argument of x, varying in the sector V, c_i $(i = 0, 1, 2, \cdots)$ are independent of x. The character of the regions of convergence, of absolute convergence and of uniform convergence are determined. The character of the points on the boundary of the region of convergence is determined, except for an isolated set of points. The character of these isolated points is determined except in very special cases.

A necessary and sufficient condition that $Y(x)$ shall converge is determined. The uniqueness of expansion of functions in terms of $Y(x)$ is determined under special conditions.

19. In this paper, Professor Lunn shows that the general relations in thermodynamics can be arranged and the notations adjusted so as to show a complete duality analogous to the similar principles already known in geometry, mechanics and electromagnetism. Energy is dual to entropy and temperature to its own reciprocal, and a corresponding ranging in pairs occurs throughout the range of quantities characteristic of the thermodynamic properties of material systems.

As illustrations of the working of the principle various applications are suggested to the dual representation of physical relations and the dual interpretations of mathematical relations, to the attainment of symmetry of notation, to the transformation of the thermodynamic potentials, and to the criteria of equilibrium and stability.

20. In this paper Professor Borger proves, by using the Denjoy integral, that if the two functions $U(x, y)$, $V(x, y)$ of the real variables x, y are continuous in these variables jointly and possess finite partial derivatives of the first order, satisfying the Cauchy-Riemann differential equations at each point of the simply connected closed region R, then U and V are analytic functions of these variables.

An application to the theory of functions of a complex variable shows that if $W = U(x, y) + iV(x, y)$ and if U and V satisfy the conditions above referred to, then W' exists and is analytic.

21. The generalized Kummer surface is, for the genus three, a 3-way, M_3, in S_7 obtained by equating the coordinates to

linearly independent theta functions of the second order and zero characteristic in three variables. Professor Coble shows that M_3 is the double manifold on a quartic locus Q_6 in S_7. The fifteen coefficients of Q_6, which are modular functions, satisfy a system of 63 cubic relations. On the other hand it is well known that seven points in a plane define a ternary quartic with isolated double tangents. Professor Coble has shown that the simplest linear system of invariants of degree three in the coordinates of each of the seven points has fifteen linearly independent members. These also satisfy a system of 63 cubic relations and are shown to be identical with the coefficients of Q_6 above. The modular group induces on these coefficients a group G of linear transformations. The invariants of G are invariants of the ternary quartic, which can be expressed either irrationally in terms of the seven points or transcendentally in terms of the theta moduli.

22. By combining some theorems of Schottky with some results of his own, Professor Coble proves the following theorems concerning the Cayley symmetroid (a certain ten-nodal quartic surface with nine absolute constants): (1) The coordinates of the ten nodes of the symmetroid can be expressed uniformly by means of abelian modular functions of genus four. (2) Under integer linear transformation of the periods the symmetroid is algebraically transformed by *regular* Cremona transformation. (3) In particular those period transformations which are congruent to identity modulo 2 leave the symmetroid unaltered.

It thus appears that the ten nodes of the symmetroid are related to the abelian modular functions of genus four as the eight base points of a net of quadrics to the functions of genus three.

23. In this paper, Professor Wilczynski thinks of a congruence of lines as the totality of the generators of a one-parameter family of ruled surfaces, basing the analytic theory upon the previously developed theory of ruled surfaces, and connecting this theory with the theory of congruences by a chain of several related theories which seem to be entirely new. The principal advantage gained for the theory of congruences consists in the new expressions found for the invariants, expressions which are elegant in form and directly

applicable to a congruence whose developables are not known. Moreover the auxiliary quantities in terms of which these invariants are expressed are interpretable in terms of the new theories of one-parameter families of ruled surfaces, and nets of ruled surfaces.

24. According to Lüroth (*Crelle's Journal*, volume 68, pages 185–190) the analytic method of investigating problems of the sort here considered, as outlined by him and suggested by a paper of Clebsch, is too difficult on account of the complicated relations between the coefficients of the given and required elements and forms. Making use of a purely geometric method, Lüroth solves a number of problems in which the curves are conics.

Professor Emch shows that these and more general problems of this type may be solved by a relatively simple analytical method which consists chiefly in the utilization of a singular null system of the following type:

$$(1)\qquad \begin{aligned} \rho x_1 &= \quad 0 \quad + \alpha_2 p_{12} + \alpha_3 p_{13} + \alpha_4 p_{14}, \\ \rho x_2 &= -\alpha_1 p_{12} + \quad 0 \quad + \alpha_3 p_{23} + \alpha_4 p_{24}, \\ \rho x_3 &= -\alpha_1 p_{13} - \alpha_2 p_{23} + \quad 0 \quad + \alpha_4 p_{34}, \\ \rho x_4 &= -\alpha_1 p_{14} - \alpha_2 p_{24} - \alpha_3 p_{34} + \quad 0 \quad . \end{aligned}$$

For $i = 1, 2, 3, 4$; $k = 1, 2, 3, 4$; $i \neq k$, the x_i are the coordinates of a point, the α_i those of a plane, the p_{ik} those of a straight line. The determinant of the transformation (null system = involutory correlation throughout) is

$$(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23})^2 = 0.$$

This null system is therefore singular and has the following geometric meaning. The plane (α_i) cuts the straight line (p_{ik}) in a point (x_i) whose coordinates are precisely those given by (1). The method based upon this system is believed to be new.

25. Professor Boutroux considers the following fundamental problem in the study of multiform functions satisfying differential equations of the first order. Calling $f(x, C)$ the general solution of an equation, find the values $C_1, C_2, \cdots$ of C which belong to the same function (that is, are such that $f(x, C_1)$,

etc., are branches of one and the same multiform function). The substitutions exchanging these values C_1, C_2, $\cdots$, form a discontinuous group G, and to investigate that group, we have to consider certain functions $\psi(C)$ (substituting functions). As a rule these last functions are themselves multiform functions. But some families of differential equations have the property that to build up the corresponding group G, it is not necessary to use the whole functions $\psi(C)$ but merely special branches or uniform "elements" of these functions which can be defined simply. Thus the difficulties arising from the multiform character of the functions in the problem are overcome.

As an example, one may take the functions $z(x)$ satisfying $z(dz/dx) = mz + 2(x^3 - 1)$. Choosing for C a parameter which determines z at $x = \infty$, one first sees that all the substitutions of G are of the form $\left[C, C + m\int_\Gamma z dx \right]$, where Γ is any closed path starting from $x = \infty$. Now it is possible to find two special substituting functions of this type, ψ_1, ψ_2, each of which is holomorphic in the whole C-plane limited by two straight cuts (straight lines joining two certain points to $x = \infty$). Considering the said functions exclusively in these limited fields, one can define uniformly for all values of C, two substitutions $[C, \psi_1(C)]$ and $[C, \psi_2(C)]$. These particular substitutions, combined with a third one of the form $[C, C + \text{constant}]$ will be sufficient, if properly chosen, to build up the total group G.

26. The coefficient of heat conductivity of most fluids, determined from observations made under controlled conditions, in which motion is eliminated, is very small. Moreover, in order to apply the results of the classical theory of the flow of heat to fluids, they must be regarded as solids. In this paper, Dr. McEwen presents the derivation of differential equations of the flow of heat in a fluid due to a convective, or alternating circulation of small fluid volumes, both when the average fluid velocity across a given surface is zero, and when it has an appreciable value. Two sets of equations are derived; one for the relatively cold elements, the other for the warm elements. The temperature that would be indicated by a thermometer depends upon both the warm and cold elements since they are intermixed. The measurable tempera-

ture is therefore a mean of these two, and a proper combination of the first two sets of equations results in a single set for determining this temperature.

27. The usual method of deducing frequency curves from assumptions as to the manner in which elementary errors or deviations arise often requires a larger number of measurements than are available in practical applications. Moreover, the difficulty of determining and handling the proper frequency curve except in special cases in which the Gaussian, or normal law holds, restricts the use of such deductive methods, and often results in erroneous applications of the simple normal law. By arranging the observations in their order of magnitude, and regarding two groups, those above the average and those below, as samples of corresponding groups of an indefinitely large sample, Dr. McEwen reduces the problem of the precision of averages to the simple one of variation in proportions. Regarding the observed average of the whole series as the best estimate of that of an indefinitely large sample, and changing the proportion of observations above and below the average, the probability of these variations in proportions corresponds to that of the resulting variation in the average computed from the new group values and the observed constant proportion. A frequency distribution of averages thus computed gives directly the probability that the true average falls within given limits. From similar determinations of the frequency distribution of the averages of two sets of observations, the probability can be found that the true values differ by an amount exceeding any specified value. A modification of this "group proportion" method not requiring the determination of the frequency distributions serves to estimate rapidly the probability that an observed difference between two averages arose by chance. The above results apply to any number of observations, and to any type of frequency distribution of the observations.

O. D. KELLOGG, *Secretary of the Southwestern Section.*
ARNOLD DRESDEN, *Secretary of the Chicago Section.*

PONCELET POLYGONS IN HIGHER SPACE.

PROFESSOR ALBERT A. BENNETT.

(Read before the American Mathematical Society December 30, 1919.)

LET there be given a linear projective space of $2n$ dimensions. A point of the space may be denoted by P and its dual figure by P'. Thus a P' is a linear space of $2n-1$ dimensions.

The totality of P's in the space is infinity to the order $2n$, and the totality of P''s is of course of this same order. We shall select from these totalities a Q_n and a Q_n' respectively, general quadratic loci of infinity to the order n of elements, where Q_n consists of P's, and Q_n' of P''s.

For Q_n and Q_n' not in specialized relation to each other we have a two-two correspondence of the following form: Each P of Q_n meets two P''s of Q_n', and each P' of Q_n' meets two P's of Q_n. Starting with any point of Q_n, a succession of points of Q_n is determined, where furthermore consecutive points of the sequence may be joined by lines. The succession of lines forms then a single "broken line" as this term is used in projective geometry. It may or may not happen that the broken line closes into a polygon. Except for degenerate cases corresponding to coincident P's or P''s, and it being supposed that Q_n and Q_n' are not degenerate, it may be proved that the closure of the broken line is determined by the relative positions of Q_n and Q_n' and is independent of the element selected as initial.

This may be called a theorem of Poncelet polygons in higher spaces. For $n = 1$, the theorem is the usual one.

It should be emphasized that the case for $n > 1$ is not the logical equivalent of the case for $n = 1$, since there are n independent parameters in any case. The proof of the theorem is immediate by reference to general theorems on algebraic correspondences or to theta functions, the quadratics Q_n and Q_n' determining theta functions of genus n, and affording one of the simplest illustrations of their character.

A second generalization and one which applies to three-space is to spaces of $2n-1$ dimensions generally, $n > 1$, the P, P', Q_n, Q_n' being as above. Any P' of Q_n' may be viewed

as a $(2n-1)$-space tangent to the n-dimensional quadratic cone K_n' of $(n-2)$-spaces also represented as Q_n'. While a P' of Q_n' meets Q_n in a conic, the two $(2n-2)$-spaces, L', tangent to K_n' and contained in P', which are also tangent to Q_n, determine two points of tangency on Q_n. This correspondence is again two-two, and for it the same theorem holds. The case $n=2$ leads to the study of Kummer's surface and the theorem is in substance familiar in this case. Cf. Hudson, Kummer's Quartic Surface, Cambridge, 1905, page 196, and Zeuthen, Lehrbuch der abzählenden Methoden der Geometrie, Leipzig, 1914, page 276.

WASHINGTON, D. C.,
November, 1919.

ON THE RECTIFIABILITY OF A TWISTED CUBIC.

BY DR. MARY F. CURTIS.

(Read before the American Mathematical Society December 31, 1919.)

IF the space curve

$$x_i = a_i t^n + b_i t^{n-1} + \cdots + k_i t + l_i \quad (i = 1, 2, 3) \tag{1}$$

is a helix, it is algebraically rectifiable. For if it is a helix, it makes with a fixed direction a constant angle and $\sqrt{x'|x'} = (x'|\alpha)$,* where α_1, α_2, α_3 are constants, not all zero; then the arc

$$s = \int_{t_0}^{t} \sqrt{x'|x'}\, dt \tag{2}$$

is an integral rational function of t, not identically zero, and the curve (1) is algebraically rectifiable.

It is not, however, in general true, that if (1) is algebraically rectifiable, it is a helix. It will be true, provided (2) is an algebraic function only when $(x'|x')$ is a perfect square of the form $(x'|\alpha)^2$. This condition is fulfilled in the case of the twisted cubic:

$$x_1 = at, \quad x_2 = bt^2, \quad x_3 = ct^3, \quad abc \neq 0, \tag{3}$$

* If $a : (a_1, a_2, a_3)$ and $b : (b_1, b_2, b_3)$ are two triples, then by $(a \mid b)$ we mean their inner product: $a_1b_1 + a_2b_2 + a_3b_3$.

and hence, as I pointed out in a note on page 87 of volume 25 (1918) of this BULLETIN, the cubic (3) is algebraically rectifiable when and only when it is a helix.

As a generalization of this result Professor Tsuruichi Hayashi, in the November number of the current volume of the BULLETIN, has for the twisted cubic

$$(4)\quad x_i = a_i t^3 + b_i t^2 + c_i t + d_i \qquad (i = 1, 2, 3, \quad |abc| \neq 0),$$

which, by a change of parameter and axes, he writes in the form

$$(5)\quad x_1 = a_1 t^3 + c_1 t, \quad x_2 = a_2 t^3 + b_2 t^2, \quad x_3 = a_3 t^3, \quad a_3 b_2 c_1 \neq 0,$$

the theorem: The conditions

$$(6)\qquad a_2 = 0, \qquad 9a_3^2 c_1^2 = 12 a_1 c_1 b_2^2 + 4 b_2^4,$$

are necessary and sufficient both that (5) be a helix and that (5) be algebraically rectifiable.

This theorem, as stated, is not true. The cubic (5) may be algebraically rectifiable if not a helix, for its arc (2), where

$$(x'|x') = 9(a|a)t^4 + 12 a_2 b_2 t^3 + (6 a_1 c_1 + 4 b_2^2) t^2 + c_1^2,$$

is an algebraic function, not only when $(x'|x')$ is a perfect square but also when $(x'|x')$ is a cubic with a multiple root, namely, when

$$(7)\quad a_2 \neq 0, \quad (a|a) = 0, \quad 2(3 a_1 c_1 + 2 b_2^2)^3 + 243 a_2^2 b_2^2 c_1^2 = 0.$$

Hence Professor Hayashi's theorem is true for the cubics (5) only if those for which (7) holds are excluded.

Since the cubics (5) for which (7) holds are imaginary, we may say: *A real twisted cubic* (5) *is algebraically rectifiable, when and only when it is a helix.*

It is interesting to note that the facts for the cubics given by (5) do not quite correspond to those for the cubics given by (4), from which (5) was deduced, because, in the reduction of (4) to (5) certain imaginary cubics given by (4) have been excluded; among them are the helices, for which $(x'|x')$ for (4) reduces to the form: $(kt^2 + lt)^2 \not\equiv 0$, and the algebraically rectifiable cubics, not helices, obtained when $(x'|x')$ for (4) is of the form: $mt + n$, $m \neq 0$. But, as these curves are all imaginary, the theorem just stated for the cubics (5) holds equally well for the cubics (4).

In order that the general twisted cubic (4) be algebraically rectifiable, it is sufficient, but not necessary, that it be a helix. By demanding algebraic rectifiability of a particular type, we may obtain a condition in terms of the curvature and torsion of (4), which is both necessary and sufficient:

The general twisted cubic (4) *may be represented by equations of the form* (4), *where t is its arc, when and only when it is of constant curvature and torsion.*

For a necessary and sufficient condition that (4) may be so represented is that $(x' | x')$ is a constant, not zero, and this in turn can be shown to be necessary and sufficient that (4) is a curve of constant curvature and torsion.

WELLESLEY COLLEGE,
November 15, 1919.

NOTE ON LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER WHOSE SOLUTIONS SATISFY A HOMOGENEOUS QUADRATIC IDENTITY.

BY DR. C. N. REYNOLDS, JR.

(Read before the American Mathematical Society December 31, 1919.)

IN this paper I shall prove that if a given homogeneous linear differential equation of the fourth order has four linearly independent real solutions which satisfy a non-singular homogeneous quadratic identity, then it may be reduced to a form which may be said to be self-adjoint with respect to the third row of the wronskian of any four linearly independent solutions. I shall then evaluate the signature of the quadratic identity in terms of the coefficients of the reduced equation and derive several theorems concerning the zeros of solutions of such equations.

Brioschi* has shown that if $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ are four linearly independent real solutions of the equation

$$(1) \qquad \begin{aligned} &p_0(x)y^{\text{IV}}(x) + 4p_1(x)y'''(x) + 6p_2(x)y''(x) \\ &\quad + 4p_3(x)y'(x) + p_4(x)y(x) = 0 \quad (p_0(x) \neq 0), \end{aligned}$$

* "Les invariants des équations différentielles linéaires," *Acta Mathematica* (1890), vol. 14, p. 233.

with real coefficients which together with a sufficient number of their derivatives are continuous, and if

$$\sum_{i=1}^{4}\sum_{j=1}^{4} A_{ij}y_i(x)y_j(x) \equiv 0 \qquad (A_{ij} = A_{ji}), \tag{2}$$

where not all of the A's are zero; then, by means of a suitable change of variables we can reduce our equations (1) and (2) to the form

$$\begin{aligned} &v^{IV}(z) + 6q_2(z)v''(z) + 9q_2'(z)v'(z) \\ &\quad + [3q_2''(z) + C]v(z) = 0 \qquad (z = f(x), \qquad f'(x) \neq 0), \end{aligned} \tag{3}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} A_{ij}v_i(z)v_j(z) \equiv 0 \qquad (A_{ij} = A_{ji}), \tag{4}$$

where

$$\Delta \equiv \begin{vmatrix} v_1(z), & v_2(z), & v_3(z), & v_4(z) \\ v_1'(z), & v_2'(z), & v_3'(z), & v_4'(z) \\ v_1''(z), & v_2''(z), & v_3''(z), & v_4''(z) \\ v_1'''(z), & v_2'''(z), & v_3'''(z), & v_4'''(z) \end{vmatrix} \equiv (v_1, v_2', v_3'', v_4''')$$

is a constant different from zero. Cels* has called the equation satisfied by the cofactors of $v_i''(z)$ $(i = 1, 2, 3, 4)$ in Δ the "equation adjoint to (3) with respect to the third row of Δ." By repeated differentiation of (v_1, v_2', v_3''') we find that it satisfies (3). This equation may therefore be said to be self-adjoint with respect to the third row of Δ.

If we now define $v_i(z)$ by means of the boundary conditions

$$\begin{aligned} v_i^{(i-1)}(\zeta) &= 1, \qquad (i, k = 1, 2, 3, 4; \\ v_i^{(k-1)}(\zeta) &= 0, \qquad k \neq i) \end{aligned}$$

then $v(z)$, any solution of (3), may be expressed in the form

$$v(z) = v(\zeta)v_1(z) + v'(\zeta)v_2(z) + v''(\zeta)v_3(z) + v'''(\zeta)v_4(z).$$

In particular, we can so express the cofactors of $v_i''(z)$ in Δ:

$$\begin{aligned} (v_2, v_3', v_4''') &= \qquad\qquad\qquad v_3, \\ -(v_1, v_3', v_4''') &= -v_2 \qquad\qquad + 6q_2(\zeta)v_4, \\ (v_1, v_2', v_4''') &= v_1 \qquad - 6q_2(\zeta)v_3 \qquad - 3q_2'(\zeta)v_4, \\ -(v_1, v_2', v_3''') &= 6q_2(\zeta)v_2 - 3q_2'(\zeta)v_3 - [36q_2^2(\zeta) - C]v_4. \end{aligned}$$

* "Sur les équations différentielles linéaires ordinaires," *Annales Scient. de l'Ecole Normale Supérieure* (1891), vol. 8, 3d ser., p. 341.

Substituting these values of our three rowed determinants in the expansion of the determinant

$$(v_1, v_2', v_3, v_4''') \equiv 0$$

in terms of the elements of its third row, we have

$$[v_1 + (-3q_2(\zeta) + \tfrac{1}{2})v_3 - 3q_2'(\zeta)v_4]^2 - [v_1 + (-3q_2(\zeta) - \tfrac{1}{2})v_3 - 3q_2'(\zeta)v_4]^2 - [v_2 - 6q_2(\zeta)v_4]^2 + Cv_4^2 \equiv 0.$$

Therefore, *the signature of the quadratic form in* (4) *is zero if* C *is positive, two if* C *is negative.*

If $v_1(z)$ and $v_2(z)$ are any solutions of (3) then

$$S(v_1, v_2) \equiv (v_1''' + 6q_2v_1' + 3q_2'v_1)(v_2''' + 6q_2v_2' + 3q_2'v_2) + C(v_1''v_2 - v_1'v_2' + v_1v_2'' + 6q_2v_1v_2)$$

is constant since

$$\frac{d}{dz}S(v_1,v_2) \equiv 0.$$

If $v(\zeta) = v'(\zeta) = 0$ then $S(v, v) = v'''^2(\zeta) \geqq 0$. Therefore, no solution $v(z)$ of (3) for which $S(v, v) < 0$ can have a multiple zero. Similarly we find that

If $C < 0$ and $S(v, v) < 0$ then $v(z)$ has no zeros.

If $C[S(v_1, v_1)S(v_2, v_2) - S^2(v_1, v_2)] > 0$ then $v_1(z)$ and $v_2(z)$ have no zeros in common.

Brioschi has shown that if $C > 0$ then any solution of (3) is a bilinear form in the solutions of

$$u''(z) + [\tfrac{3}{2}q_2(z) + r]u(z) = 0, \qquad 4r^2 = C, \qquad r > 0, \tag{5}$$

$$w''(z) + [\tfrac{3}{2}q_2(z) - r]w(z) = 0. \tag{6}$$

If $u_1(z)$, $u_2(z)$, $w_1(z)$ and $w_2(z)$ are defined by the boundary conditions:

$$u_i^{(i-1)}(\zeta) = 1, \qquad w_i^{(i-1)}(\zeta) = 1, \qquad (i, k = 1, 2;$$

$$u_i^{(k-1)}(\zeta) = 0, \qquad w_i^{(k-1)}(\zeta) = 0, \qquad k \neq i)$$

then $v(z)$, any solution of (3), may be written in the form

$$v(z) = v(\zeta)u_1(z)w_1(z)$$

$$-\frac{1}{4r}[v''' + (6q_2 - 2r)v' + 3q_2'v]_{x=\zeta}u_1(z)w_2(z)$$

$$+\frac{1}{4r}[v''' + (6q_2 + 2r)v' + 3q_2'v]_{x=\zeta}u_2(z)w_1(z)$$

$$+\tfrac{1}{2}[v''(\zeta) + 3q_2(\zeta)v(\zeta)]u_2(z)w_2(z).$$

$S(v, v) = 0$ is therefore the necessary and sufficient condition that $v(z)$ be factorable as the product of a solution of (5) by a solution of (6). Therefore, if $S(v, v) = 0$ the number of zeros of $v(z)$ in a given closed interval (a, b) will be limited by Sturm's theorems concerning the zeros of solutions of (5) and (6). If, for example, the equations (5) and (6) have k_1 and k_2 for their respective indices of oscillation in (a, b) then every solution of (5) [(6)] will have k_1 or $k_1 + 1$ [k_2 or $k_2 + 1$] zeros in the interval (a, b), and every solution, $v(z)$, of (3) for which $S(v, v) = 0$ will have $k_1 + k_2$, $k_1 + k_2 + 1$, or $k_1 + k_2 + 2$ zeros in (a, b). If, in particular, $q_2(z) < 0$ and $0 < C < 9q_2^2(z)$ throughout (a, b) then the coefficients of $u(z)$ and $w(z)$ in (5) and (6) will be negative throughout (a, b) and $k_1 = k_2 = 0$.* Therefore, *if* $q_2(z) < 0$ *and* $0 < C < 9q_2^2(z)$ *throughout* (a, b) *then no solution of* (3) *for which* $S(v, v) = 0$ *has more than two zeros in* (a, b), *and some such solutions have no zeros in* (a, b).

WESLEYAN UNIVERSITY,
MIDDLETOWN, CONN.

AN ACKNOWLEDGMENT OF PRIORITY.

BY PROFESSOR A. A. BENNETT.

IN the July, 1919, number of the BULLETIN (volume 25, page 455), I gave a solution condensed to about two pages, of a problem proposed by the Philosophical Faculty of the University of Berlin as a prize problem. My attention has been called to the fact that this question is completely handled in an analogous manner by Harold Hilton, in his book on "Homogeneous Linear Substitutions," 1914, in Chapter VI, section 8. Similar results are given in the same chapter for

* See Bôcher's Méthodes de Sturm, Paris, 1917, p. 52.

symmetric and Hermitian matrices. This book, while familiar to the American public, was not available to me at the time, and I did not realize that the problem proposed had been completely settled previous to the occurrence of the war.

February 9, 1920.

DICKSON'S HISTORY OF THE THEORY OF NUMBERS.

On page 130 of my review of Dickson's History I speak of two pages of titles at the end of Chapter XII as "not reported on." This is an error. With the exception of certain ones marked with an asterisk which give papers not obtainable by the author the content of the papers is sufficiently indicated, the papers not being of importance to warrant more detailed report.

The phrase "list of references," on the top of page 132, is perhaps misleading. Of course the book is in no sense to be compared with the useless lists of titles of papers which the compiler may or may not have glanced at. My intention in the paragraph was to bring out the distinction which modern historians seem to make between a list of events and the relations and connections between events.

D. N. Lehmer.

NOTES.

The December number (volume 21, number 2) of the *Annals of Mathematics* contains the following papers: "A property of cyclotomic integers and its relation to Fermat's last theorem," by H. S. Vandiver; "Surfaces of rotation in a space of four dimensions," by C. L. E. Moore; "The circle nearest to n given points, and the point nearest to n given circles," by J. L. Coolidge; "Singular solutions of differential equations of the second order," by E. M. Coon and R. L. Gordon; "Note on a class of integral equations of the second kind," by C. E. Love; "Concerning sense on closed curves in

non-metrical plane analysis situs," by J. R. KLINE; "On the theory of summability," by GLENN JAMES; "On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable," by L. L. SILVERMAN.

THE Mathematical association of Japan for secondary education issued the first number of volume 1 of its *Journal* under date of April, 1919. It is published by M. KABA, of Tokyo, as chief editor, assisted by M. KURODA, M. WATANABE, S. NAGAO and H. FURUKAWA. Professor T. HAYASHI is president of the Association.

IT is announced that a new quarterly devoted to the mathematics of finance, the *Giornale di Matematica Finanziaria*, will be published under the auspices of the higher institutes of commerce and the chambers of commerce of Turin and Genoa. The business offices are at 73, Corso Vittorio Emmanuele, Turin.

THE Paris Academy of sciences has awarded its Bordin prize for 1919 to Professor S. LEFSCHETZ, of the University of Kansas, for his memoir "On certain invariant numbers of algebraic varieties, with application to abelian varieties." The Academy announces the following subject for its Grand prize, to be awarded in 1922: The determination of extensive classes of surfaces by given properties of their geodesics considered in ordinary space.

PROFESSOR C. SERVAIS, of the University of Ghent, has been elected an active member, and Professor T. DE DONDER, of the University of Brussels, a corresponding member of the Belgian academy of sciences.

PROFESSORS J. PLEMELJ, of the University of Czernovitch, and R. SUPPANTSCHITSCH, of the technical school of Vienna, have been appointed professors of mathematics at the reorganized University of Lioubliana (Laibach), Jugoslavia.

MR. J. H. JEANS, of Cambridge, formerly professor of mathematics at Princeton University, has been nominated as secretary of the Royal society of London. Professor G. N.

WATSON, of the University of Birmingham, has been elected a fellow of the society.

SIR J. J. THOMSON, master of Trinity College, Cambridge, who recently resigned the Cavendish professorship of experimental physics, has been elected to the newly established professorship of physics.

CAPT. L. L. BURCHNALL, scholar of Christ Church, Oxford, has been appointed lecturer in mathematics at the University of Durham.

DR. R. J. T. BELL, of the University of Glasgow, has been appointed to the chair of pure and applied mathematics in the University of Otago, New Zealand.

PROFESSORS N. B. MACLEAN and N. R. WILSON have returned to the University of Manitoba, from active service in France.

PROFESSOR E. W. BROWN delivered a lecture on the history of mathematics before the Gamma Alpha fraternity of Yale University on February 26. This is the first of a series on the history of the several sciences. The Astronomical Society of the Pacific has awarded its Bruce Medal for 1920 to Professor BROWN.

AT Yale University, assistant professor W. R. LONGLEY has been promoted to a professorship and Mr. J. K. WHITTEMORE to an assistant professorship of mathematics.

PROFESSOR P. P. BOYD, of the University of Kentucky, has been elected president of the Kentucky academy of sciences.

BOSTON University has concluded an arrangement for the exchange of professorships in mathematics for the college year 1920–1921 with Tsing Hua College, Peking, China. Professor R. E. BRUCE, chairman of the department in Boston University, will exchange with Professor A. H. HEINZ, of Tsing Hua.

ASSISTANT professor K. P. WILLIAMS, of Indiana University, has returned from military service and been promoted to an associate professorship of mathematics.

DR. C. C. CAMP, of the University of Illinois, has been appointed assistant professor of mathematics at Iowa State College.

ASSISTANT professor ARTHUR RAMSEY has been promoted to a professorship of mathematics at Grove City College.

MISS OTTILIA W. DUEKER has been appointed professor of mathematics and dean of women at the Friends' University, Wichita, Kansas.

DR. J. S. MILLER, of Emory and Henry College, has been appointed professor of mathematics at Hampden-Sidney College.

MR. J. E. DOTTERER has been appointed professor of mathematics and physics at Manchester (Indiana) College.

ASSISTANT professor C. P. SOUSLEY, of Pennsylvania State College, has been promoted to an associate professorship of mathematics.

MR. V. G. GROVE, of Cornell University, has been appointed assistant professor of mathematics at the Michigan Agricultural College.

AT Cornell University, the following appointments to instructorships in mathematics have been made: Dr. H. C. M. MORSE, of Harvard University, Mr. P. A. FRALEIGH, of Dartmouth College, and Mr. F. M. LUFKIN.

DR. W. H. WILSON, of the Massachusetts Institute of Technology, has been appointed instructor in mathematics at the University of Iowa.

THE following instructors in mathematics have been appointed at the University of Wisconsin: C. D. EHRMAN, I. W. TRAYLER, J. E. DAVIS, M. INGRAHAM, F. E. RECHART, Miss E. M. F. FELTGES, and Miss B. ALBRIGHT.

PROFESSOR H. G. ZEUTHEN, of the University of Copenhagen, died January 6, 1920, at the age of eighty years.

He spent the summer of 1919 in writing a seventy-page paper on the origin of algebra.

REV. C. J. BORGMEYER, professor of mathematics and astronomy at St. Louis University, died December 6, 1919, at the age of fifty-eight years.

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